Scalar Field Cosmologies with Barotropic Matter: Models of Bianchi class B

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We investigate in detail the qualitative behaviour of the class of Bianchi type B spatially homogeneous cosmological models in which the matter content is composed of two non–interacting components; the first component is described by a barotropic fluid having a gamma-law equation of state, whilst the second is a non–interacting scalar field ϕ with an exponential potential $V(\phi) = \Lambda e^{k\phi}$. In particular, we study the asymptotic properties of the models both at early and late times, paying particular attention on whether the models isotropize (and inflate) to the future, and we discuss the genericity of the cosmological scaling solutions.

I. INTRODUCTION

Scalar field cosmology is of importance in the study of the early Universe and particularly in the investigation of inflation (during which the universe undergoes a period of accelerated expansion [1,2]). One particular class of inflationary cosmological models are those with a scalar field and an exponential potential of the form $V(\phi) = \Lambda e^{k\phi}$, where Λ and k are non-negative constants. Models with an exponential scalar field potential arise naturally in alternative theories of gravity, such as, for example, scalar-tensor theories.

Scalar-tensor theories of gravitation, in which gravity is mediated by a long-range scalar field in addition to the usual tensor fields present in Einstein's theory, are perhaps the most natural alternatives to general relativity (GR). In the simplest Brans-Dicke theory of gravity (BDT; [3]), a scalar field, ϕ, with a constant coupling parameter ω_0, acts as the source for the gravitational coupling. More general scalar-tensor theories have a non-constant parameter, $\omega(\phi)$, and a non-zero self-interaction scalar potential, $V(\phi)$. Observational limits on the present value of ω_0 need not constrain the value of ω at early times in more general scalar-tensor theories (than BDT). Hence, more recently there has been greater focus on the early Universe predictions of scalar-tensor theories of gravity, with particular emphasis on cosmological models in which the scalar field acts as a source for inflation [2,4]. BDT (and other theories of gravity, such as, for example, more general scalar-tensor theories and quadratic Lagrangian theories and also theories undergoing dimensional reduction to an effective four-dimensional theory [5]), are known to be conformally equivalent to general relativity plus a scalar field having exponential-like potentials [5,6].

Scalar-tensor theory gravity is currently of particular interest since such theories occur as the low-energy limit in supergravity theories from string theory and other higher-dimensional gravity theories [7]. Lacking a full non-perturbative formulation which allows a description of the early Universe close to the Planck time, it is necessary to study classical cosmology prior to the GUT epoch by utilizing the low-energy effective action induced by string theory. To lowest order in the inverse string tension the tree-level effective action in four-dimensions for the massless fields includes the non-minimally coupled graviton, the scalar dilaton and an antisymmetric rank-two tensor, hence generalizing GR (which is presumably a valid description at late, post-GUT, epochs) by including other massless fields; hence the massless bosonic sector of (heterotic) string theory reduces generically to a four-dimensional scalar-tensor theory of gravity. As a result, BDT includes the dilaton-graviton sector of the string effective action as a special case ($\omega = -1$) [7]. String cosmology has recently been investigated by various authors [8], and, in particular, [9] presented a qualitative analysis for spatially flat, isotropic and homogeneous cosmologies derived from the string effective action when a cosmological constant term is included. A discussion of how exponential potentials arise in effective four-dimensional theories (in the so-called conformal Einstein frame) after dimensional reduction from higher-dimensional theories such as string theory and M-theory is given in [10].

A number of authors have studied scalar field cosmological models with an exponential potential within GR. Homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models were studied by Halliwell [5] using phase-

plane methods (see also [2]). Homogeneous but anisotropic models of Bianchi types I and III (and Kantowski-Sachs models) have been studied by Burd and Barrow [11] in which they found exact solutions and discussed their stability. Lidsey [12] and Aguirregabiria et al. [13] found exact solutions for Bianchi type I models, and in the latter paper a qualitative analysis of these models was also presented. Bianchi models of types III and VI were studied by Feinstein and Ibáñez [14], in which exact solutions were found. A qualitative analysis of Bianchi models with $k^2 < 2$, including standard matter satisfying various energy conditions, was completed by Kitada and Maeda [15]. They found that the well-known power-law inflationary solution is an attractor for all initially expanding Bianchi models (except a subclass of the Bianchi type IX models which will recollapse).

The governing differential equations in spatially homogeneous Bianchi cosmologies containing a scalar field with an exponential potential exhibit a symmetry [16], and when appropriate expansion- normalized variables are defined, the governing equations reduce to a dynamical system, which was studied qualitatively in detail in [17]. In particular, the question of whether the spatially homogeneous models inflate and/or isotropize, thereby determining the applicability of the so-called cosmic no-hair conjecture in homogeneous scalar field cosmologies with an exponential potential, was addressed. The relevance of the exact solutions (of Bianchi types III and VI) found by Feinstein and Ibáñez [14], which neither inflate nor isotropize, was also considered. In a follow up paper [18] the isotropization of the Bianchi VII$_h$ cosmological models possessing a scalar field with an exponential potential was further investigated; in the case $k^2 > 2$, it was shown that there is an open set of initial conditions in the set of anisotropic Bianchi VII$_h$ initial data such that the corresponding cosmological models isotropize asymptotically. Hence, scalar field spatially homogeneous cosmological models having an exponential potential with $k^2 > 2$ can isotropize to the future. However, in the case of the Bianchi type IX models having an exponential potential with $k^2 > 2$ the result is different in that typically expanding Bianchi type IX models do not isotropize to the future; the analysis of [19] indicates that if $k^2 > 2$, then the model recollapses.

Recently cosmological models which contain both a perfect fluid description of matter and a scalar field with an exponential potential have come under heavy analysis. One of the exact solutions found for these models has the property that the energy density due to the scalar field is proportional to the energy density of the perfect fluid, hence these models have been labelled scaling cosmologies [20,21]. With the discovery of these scaling solutions, it has become imperative to study spatially homogeneous Bianchi cosmologies containing a scalar field with an exponential potential and an additional matter field consisting of a barotropic perfect fluid. The scaling solutions studied in [20,21], which are spatially flat isotropic models in which the scalar field energy density tracks that of the perfect fluid, are of particular physical interest. For example, in these models a significant fraction of the current energy density of the Universe may be contained in the scalar field whose dynamical effects mimic cold dark matter.

In [22] the stability of these cosmological scaling solutions within the class of spatially homogeneous cosmological models with a perfect fluid subject to the equation of state $p_\gamma = (\gamma - 1)\rho_\gamma$ (where γ is a constant satisfying $0 < \gamma < 2$) and a scalar field with an exponential potential was studied. It is known that the scaling solutions are late-time attractors (i.e., stable) in the subclass of flat isotropic models [20,21]. In [22] it was found that that the scaling solutions are stable (to shear and curvature perturbations) in generic anisotropic Bianchi models when $\gamma < 2/3$. However, when $\gamma > 2/3$, and particularly for realistic matter with $\gamma \geq 1$, the scaling solutions are unstable; essentially they are unstable to curvature perturbations, although they are stable to shear perturbations. Although these solutions are unstable, since they correspond to equilibrium points of the governing dynamical system, the Universe model can spend an arbitrarily long time near these scaling solutions, and hence they may still be of physical importance.

In addition to the scaling solutions described above, curvature scaling solutions and anisotropic scaling solutions are also possible. In [23] homogeneous and isotropic spacetimes with non-zero spatial curvature were studied in detail and three possible asymptotic future attractors in an ever-expanding universe were found. In addition to the zero-curvature power-law inflationary solution and the zero-curvature scaling solution alluded to above, there is a solution with negative spatial curvature where the scalar field energy density remains proportional to the curvature, which also acts as a possible future asymptotic attractor. In [24] spatially homogeneous models with a perfect fluid and a scalar field with an exponential potential were also studied and the existence of anisotropic scaling solutions was also discovered; the stability of these anisotropic scaling solutions within a particular class of Bianchi type models was discussed.

The purpose of this paper is to comprehensively study the qualitative properties of spatially homogeneous models with a barotropic fluid and a non-interacting scalar field with an exponential potential in the class of Bianchi type B models (except for the exceptional case Bianchi VI$_{-1/9}$), using the Hewitt and Wainwright formalism [25,26]. In particular, we shall study the generality of the scaling solutions. The paper is organized as follows. In section II we define the governing equations, which are modified from those developed in [25], and discuss the invariant sets and the existence of monotonic functions. In section III, we classify and list all of the equilibrium points, and their local stability is discussed in section IV. We give a detailed analysis, including heteroclinic orbits, for a subset of Bianchi type VI$_h$ models in section V. We leave conclusions and discussion for section VI.

II. THE EQUATIONS

We shall assume that the matter content is composed of two non-interacting components. The first component is a separately conserved barotropic fluid with a gamma-law equation of state, i.e., $p = (\gamma - 1)\mu$, where γ is a constant with $0 \leq \gamma \leq 2$, while the second is a noninteracting scalar field ϕ with an exponential potential $V(\phi) = \Lambda e^{k\phi}$, where Λ and k are positive constants (we use units in which $8\pi G = c = 1$). By non-interacting we mean that the energy-momentum of the two matter components will be separately conserved.

The state of any Bianchi type B model with the above matter content can be described by the evolution of the variables

$$\left(H, \sigma_+, \tilde{\sigma}, \delta, \tilde{a}, n_+, \dot{\phi}, \phi\right) \in \mathbb{R}^8, \tag{2.1}$$

where the evolution of the state variables are given as equations (5.8) and (7.8) in Wainwright and Ellis [25] with the addition of the Klein-Gordon equation for the scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + kV(\phi) = 0. \tag{2.2}$$

By introducing dimensionless variables, the evolution equation for H decouples and the resulting reduced system has one less dimension [25]. Defining [25,17]

$$\begin{aligned} &\Sigma_+ = \frac{\sigma_+}{H}, \quad \tilde{\Sigma} = \frac{\tilde{\sigma}}{H^2}, \quad \Delta = \frac{\delta}{H^2}, \quad \tilde{A} = \frac{\tilde{a}}{H^2}, \\ &N_+ = \frac{n_+}{H}, \quad \Psi = \frac{\dot{\phi}}{\sqrt{6}H}, \quad \Phi = \frac{\sqrt{V(\phi)}}{\sqrt{3}H}, \quad \Omega = \frac{\mu}{3H^2}, \end{aligned} \tag{2.3}$$

the differential equations for the quantities

$$\mathbf{X} = \left(\Sigma_+, \tilde{\Sigma}, \Delta, \tilde{A}, N_+, \Psi, \Phi\right) \in \mathbb{R}^7 \tag{2.4}$$

are as follows:

$$\Sigma_+' = (q-2)\,\Sigma_+ - 2\,\tilde{N}, \tag{2.5}$$

$$\tilde{\Sigma}' = 2\,(q-2)\,\tilde{\Sigma} - 4\,\Delta\,N_+ - 4\,\Sigma_+\,\tilde{A}, \tag{2.6}$$

$$\Delta' = 2\,(q + \Sigma_+ - 1)\,\Delta + 2\,(\tilde{\Sigma} - \tilde{N})\,N_+, \tag{2.7}$$

$$\tilde{A}' = 2\,(q + 2\,\Sigma_+)\,\tilde{A}, \tag{2.8}$$

$$N_+' = (q + 2\,\Sigma_+)\,N_+ + 6\,\Delta, \tag{2.9}$$

$$\Psi' = (q-2)\,\Psi - \frac{1}{2}\,\sqrt{6}\,k\,\Phi^2, \tag{2.10}$$

$$\Phi' = (q + 1 + \frac{1}{2}\,\sqrt{6}\,k\,\Psi)\,\Phi, \tag{2.11}$$

where a prime denotes differentiation with respect to the time τ, where $dt/d\tau = H$. The deceleration parameter q is defined by $q \equiv -(1 + H'/H)$, and both $\tilde{N}$ (a curvature term) and Ω (a matter term) are obtained from first integrals:

$$q = 2\,\Sigma_+^2 + 2\,\tilde{\Sigma} + \frac{1}{2}\,(3\,\gamma - 2)\,\Omega + 2\,\Psi^2 - \Phi^2, \tag{2.12}$$

$$\tilde{N} = \frac{1}{3}\,N_+^2 - \frac{1}{3}\,l\,\tilde{A}, \tag{2.13}$$

$$\Omega = 1 - \Psi^2 - \Phi^2 - \Sigma_+^2 - \tilde{\Sigma} - \tilde{N} - \tilde{A}. \tag{2.14}$$

The evolution of Ω is given by the auxiliary equation

$$\Omega' = \Omega\,(2\,q - 3\,\gamma + 2). \tag{2.15}$$

The parameter $l = 1/h$ where h is the group parameter is equivalent to Wainwright's $\tilde{h}$ in [25]. If $l < 0$ and $\tilde{A} > 0$ then the model is of Bianchi type VI_h. If $l > 0$ and $\tilde{A} > 0$ and $N_+ \neq 0$ then the model is of Bianchi type VII_h. If

$l=0$ then the model is either Bianchi type IV or type V. If $\tilde{A}=0$ then the model is either a Bianchi type I or a Bianchi type II model.

There is one constraint equation that must also be satisfied:

$$G(\mathbf{X}) = \tilde{\Sigma}\,\tilde{N} - \Delta^2 - \tilde{A}\,\Sigma_+^2 = 0, \tag{2.16}$$

Therefore the state space is six-dimensional; the seven evolution equations (2.5)-(2.11) are subject to the constraint equation (2.16). We shall refer to the seven-dimensional state space (2.4) as the *extended* state space.

By definition $\tilde{A}$ is non-negative, which implies from equations (2.16) and (2.13) that $\tilde{\Sigma}$ and $\tilde{N}$ are also non-negative. Thus we have

$$\tilde{A} \geq 0, \qquad \tilde{\Sigma} \geq 0, \qquad \tilde{N} \geq 0. \tag{2.17}$$

In addition, from the physical constraint $\Omega \geq 0$ together with equation (2.14), we find that the state space is compact. Indeed, we have that

$$0 \leq \left\{\Sigma_+^2, \tilde{\Sigma}, \Delta^2, \tilde{A}, \tilde{N}, \Psi^2, \Phi\right\} \leq 1. \tag{2.18}$$

Since both $\tilde{A}$ and $\tilde{N}$ are bounded, we have from equation (2.13) that N_+ is bounded. In equation (2.3) we take the "positive square root". In principle, there exists negative and positive values for Φ, but from the definition (2.3) a negative Φ implies a negative H and hence $H<0$ for all time; i.e., the models are contracting. Since the system is invariant under $\Phi \to -\Phi$, without loss of generality we shall only consider $\Phi \geq 0$.

A. Invariant Sets

There are a number of important invariant sets. Recall that the state space is constrained by equation (2.16) to be a six-dimensional surface in the seven-dimensional *extended* space. Taking the constraint equation (2.16) into account we calculate the dimension of each invariant set. These invariant sets can be classified into various classes according to Bianchi type and/or according to their matter content. Some invariant sets (notably the Bianchi invariant sets) have lower-dimensional invariant subsets. Equilibrium points and orbits occuring in each Bianchi invariant set correspond to cosmological models of that Bianchi type. The notation used here has been adapted from [25]. Various lower-dimensional invariant sets can be constructed by taking the intersection of any Bianchi invariant set with the various Matter invariant sets. For example, $B(I) \cap \mathcal{M}$ is a 3-dimensional invariant set describing Bianchi type I models with a massless scalar field.

Bianchi Type	Notation	Dimension	Restrictions
Bianchi I	$B(I)$	4	$\tilde{A}=\Delta=N_+=0$
	$S(I)$	2	$\tilde{A}=\Sigma_+=\tilde{\Sigma}=\Delta=N_+=0$
Bianchi II	$B^{\pm}(II)$	5	$\tilde{A}=0,\quad N_+>0$ or $N_+<0$
	$S^{\pm}(II)$	4	$\tilde{A}=0,\quad \tilde{\Sigma}=3{\Sigma_+}^2,\quad \Delta=\Sigma_+N_+$
Bianchi IV	$B^{\pm}(IV)$	6	$l=0,\quad \tilde{A}>0,\quad N_+>0$ or $N_+<0$
Bianchi V	$B(V)$	4	$l=0,\quad \tilde{A}>0,\quad \Sigma_+=\Delta=N_+=0$
	$S(V)$	3	$l=0,\quad \tilde{A}>0,\quad \Sigma_+=\tilde{\Sigma}=\Delta=N_+=0$
Bianchi VI_h	$B(VI_h)$	6	$l<0,\quad \tilde{A}>0$
	$S(VI_h)$	4	$l<0,\quad \tilde{A}>0,\quad 3\Sigma_+^2+l\tilde{\Sigma}=0,\quad N_+=\Delta=0$
	$S^{\pm}(III)$	5	$l=-1,\quad \tilde{A}>0,\quad 3\Sigma_+^2-\tilde{\Sigma}=0,\quad \Delta=\Sigma_+N_+$
Bianchi VII_h	$B^{\pm}(VII_h)$	6	$l>0,\quad \tilde{A}>0,\quad N_+>0$ or $N_+<0$
	$S^{\pm}(VII_h)$	3	$l>0,\quad \tilde{A}>0,\quad \Sigma_+=\tilde{\Sigma}=\Delta=0,\quad N_+^2=l\tilde{A}>0$

TABLE I. *Bianchi Invariant Sets. We note that $B(I)$ and $B^{\pm}(II)$ are class A Bianchi invariant sets which occur in the closure of the appropriate higher-dimensional Bianchi type B invariant set (see Fig. 1). In addition, if l is non-negative, $N_+>0$ and $N_+<0$ define disjoint invariant sets (indicated by a superscript $\pm$ in the table). Due to the discrete symmetry $\Delta\to-\Delta$, $N_+\to-N_+$, these pairs of invariant sets are equivalent.*

Matter Content	Notation	Dimension	Restrictions
Scalar Field	$\mathcal{S}$	5	$\Omega = 0; \quad \Psi \neq 0, \quad \Phi \neq 0$
Massless Scalar Field	$\mathcal{M}$	4	$\Omega = 0; \quad \Psi \neq 0, \quad \Phi = 0$
Vacuum	$\mathcal{V}$	3	$\Omega = 0; \quad \Psi = 0, \quad \Phi = 0$
Perfect Fluid + Scalar Field	$\mathcal{FS}$	6	$\Omega \neq 0; \quad \Psi \neq 0, \quad \Phi \neq 0$
Perfect Fluid + Massless Scalar Field	$\mathcal{FM}$	5	$\Omega \neq 0; \quad \Psi \neq 0, \quad \Phi = 0$
Perfect Fluid	$\mathcal{F}$	4	$\Omega \neq 0; \quad \Psi = 0, \quad \Phi = 0$

TABLE II. *Matter Invariant Sets.*

An analysis of the dynamics in the invariant sets $\mathcal{V}$ and $\mathcal{F}$ has been presented by Wainwright and Hewitt [26]. Equilibrium points and orbits in the invariant set $\mathcal{M}$ correspond to models with a massless scalar field; i.e., scalar field models with zero potential. These models are equivalent to models with a stiff perfect fluid (i.e. $\gamma = 2$) equation of state; see [26]. Equilibrium points and orbits in the invariant set $\mathcal{FM}$ can be interpreted as representing a two–perfect–fluid model with $\gamma_2 = 2$ [27]. A partial analysis of the isotropic equilibrium points in the invariant set $\mathcal{S}$ was completed by van den Hoogen et al. [28]. We note that the so-called scaling solutions [21,29,30] are in the invariant set $\mathcal{FS}$.

The isotropic and spatially homogeneous models are found in the invariant sets $S^{\pm}(VII_h) \cup S(I)$ if $l \neq 0$, and $S(V) \cup S(I)$ if $l = 0$. In particular the zero curvature isotropic models are found in the two dimensional set $S(I)$, while the negative curvature models are found in the three-dimensional sets $S^{\pm}(VII_h)$ or $S(V)$ depending upon the value of l. See van den Hoogen et al. for a comprehensive analysis of the isotropic scaling models [23].

We note that in the invariant set $B(I)$ there exists the invariant set $\tilde{\Sigma} + \Sigma_+^2 + \Psi^2 < 1, \Delta = \tilde{A} = N_+ = \Phi = 0$, which may be directly integrated to yield

$$\tilde{\Sigma} + \Sigma_+^2 + \Psi^2 = \left[1 + \zeta e^{3(2-\gamma)\tau}\right]^{-1}, \quad \zeta = \text{constant}, \tag{2.19}$$

where τ is the time parameter. This solution asymptotes into the past towards the paraboloid $\mathcal{K}$ (section III.B.1), and asymptotes to the future towards the point $\mathrm{P}(I)$. This solution belongs to the matter invariant set $\mathcal{FM}$, asymptoting into the past towards the set $\mathcal{M}$.

B. Monotone Functions

The existence of strictly monotone functions, $W(\mathbf{X}) : \mathbb{R}^n \to \mathbb{R}$, on any invariant set, S, proves the non-existence of periodic or recurrent orbits in S and can be used to provide information about the global behaviour of the dynamical system in S. (See Theorem 4.12 in [25] for details.)

Function: $W_i(\mathbf{X})$	Derivative: $W_i'(\mathbf{X})$	Region of Monotonicity
$W_1 \equiv (1+\Sigma_+)^2 - \tilde{A}$	$W_1' = -2\,(2-q)\,W_1 + 3\,(1+\Sigma_+)\left(2\Phi^2 + (2-\gamma)\,\Omega\right)$	Monotonically approaches zero in the invariant set $\mathcal{M} \cup \mathcal{V}$.
$W_2 \equiv \dfrac{1-\Omega-\Phi^2-\Psi^2}{\Omega}$	$W_2' = -W_2(2-3\gamma) - \dfrac{1}{\Omega}\left(\Sigma_+^2 + \tilde{\Sigma}\right)$	Monotonically decreasing to zero in the set $(\mathcal{FS} \cup \mathcal{FM} \cup \mathcal{F}) \backslash S(I)$ when $0 \leq \gamma \leq 2/3$
$W_3 \equiv \tilde{\Sigma}$	$W_3' = -2\,(2-q)\,W_3 - 4(\Delta N_+ + \Sigma_+ \tilde{A})$	Monotonically decreasing to zero in the invariant sets $B(I)\backslash S(I)$ and $B(V)\backslash S(V)$.
$W_4 \equiv \dfrac{\tilde{A}^2}{N_+}$	$W_4' = 3W_4\left(q + 2\dfrac{\Sigma_+ N_+ - \Delta}{N_+}\right)$	Monotonically approaches zero in the invariant set $S^{\pm}(III)\backslash(\mathcal{S} \cup \mathcal{FS})$, when $\gamma > 2/3$.

TABLE III. *Functions, their derivatives and the sets in which they are monotonic.*

Hewitt and Wainwright found a number of monotone functions in the invariant sets of dimension less than four in the perfect fluid case (i.e., in lower-dimensional subsets of the perfect fluid invariant set) and these are summarized in an Appendix in Hewitt and Wainwright [25,26]. However, they were not able to find a monotonic function in the full perfect fluid invariant set for $2/3 < \gamma < 2$.

C. The Constraint Surface

The constraint equation $G(\mathbf{X}) = 0$ and the Implicit Function Theorem can generally be used to eliminate one of the variables at any point in the *extended* state-space provided the constraint equation is not singular there, i.e., $\mathrm{grad}(G(\mathbf{X})) \neq \mathbf{0}$. The constraint surface is singular for all points in the invariant sets $S(I)$, $B(V)$ and $S(VII_h)$ and therefore cannot be used to eliminate one of the variables (and hence reduce the dimension of the dynamical system to six).

Therefore, we cannot determine the local stability of equilibrium points in the sets $S(I)$, $B(V)$ or $S(VII_h)$ within the six-dimensional state-space, and hence we are required to determine the local stability of these equilibrium points in the *extended* space, due to the singular nature of the constraint surface. This leads to further complications because of the limited use of the Stable Manifold Theorem. If these equilibrium points are stable in the *extended* state space, then they are stable in the six-dimensional constrained surface. However, if these equilibrium points are saddles in the *extended* state-space, then one cannot easily determine the dimension of the stable manifold within the constraint surface.

III. CLASSIFICATION OF THE EQUILIBRIUM POINTS

Let us analyse the evolution equations for the matter variables, namely equations (2.10) and (2.11) and the auxiliary equation (2.15). From equation (2.15) we find that at the equilibrium points either

$$(A) \quad \Omega = 0, \tag{3.1}$$

or

$$(B) \quad q = \frac{3}{2}\gamma - 1. \tag{3.2}$$

In the scalar field case (A) there is no perfect fluid present. This is the scalar field invariant set $\mathcal{S}$. The equilibrium points and their stability will be studied in subsection III.A. These models include the massless scalar field case in which $\Phi = 0$ $(V = 0)$, but not the vacuum case $\Phi = \Psi = 0$ which will be dealt with as a subcase of the perfect fluid case (see below). The equilibrium points of case (A) include the isotropic Bianchi VII$_h$ models studied in [28].

If, on the other hand, equation (3.2) is satisfied, assuming that $\gamma < 2$ so that $q \neq 2$, from equations (2.10) and (2.11) we have that

$$(B1) \quad \Psi = 0, \Phi = 0 \tag{3.3}$$

or

$$(B2) \quad \Psi = \frac{-\sqrt{3}\gamma}{\sqrt{2}k}, \quad \Phi^2 = \frac{3\gamma(2-\gamma)}{2k^2}. \tag{3.4}$$

In case $(B1)$, in which both equations (3.1) and (3.2) are valid, there is no scalar field present. The perfect fluid subcase, which was studied by Hewitt and Wainwright [26], will be dealt with in subsection III.B. Note that from equation (2.11) $\Phi = 0$ is an invariant set, denoted $\mathcal{M}$.

The final case $(B2)$, in which equation (3.2) is valid and neither the scalar field nor the perfect fluid is absent, corresponds to the scaling solutions when $\gamma > 0$. If we define

$$\mu_\phi \equiv \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi \equiv \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{3.5}$$

then from equation (3.4) we find that

$$\gamma_\phi \equiv \frac{\mu_\phi + p_\phi}{p_\phi} = \frac{2\Psi^2}{\Psi^2 + \Phi^2} = \gamma, \tag{3.6}$$

so that the scalar field "inherits" the equation of state of the fluid. It can be shown that there are exactly three equilibrium points corresponding to scaling solutions; the flat isotropic scaling solution described in [21], and whose stability was discussed within Bianchi type VII$_h$ models in [22], and two anisotropic scaling solutions [24]. This will be further discussed in subsection III.C.

Hereafter, we shall assume that $0 < \gamma < 2$. The value $\gamma = 0$ corresponds to a cosmological constant and the model can be analyzed as a scalar field model with the potential $V = V_0 + \Lambda e^{k\phi}$ [31]. The value $\gamma = 2$, corresponding to the stiff fluid case, is a bifurcation value and will not be considered further.

A. Scalar Field Case

There are seven equilibrium points and one equilibrium set in the scalar field invariant set $\mathcal{S}$ in which $\Omega = 0$. The first three equilibrium points were given in [17] (wherein matter terms were not included): they represent isotropic models ($\Sigma_+ = \tilde{\Sigma} = \tilde{N} = \Delta = 0$):

1) $\mathrm{P}_{\mathcal{S}}(I)$: $\Sigma_+ = \tilde{\Sigma} = \Delta = \tilde{A} = N_+ = 0, \Psi = -k/\sqrt{6}, \Phi = \sqrt{1 - k^2/6}$

This equilibrium point, for which $q = -1 + k^2/2$ and which exists only for $k^2 \leq 6$, is in the Bianchi I invariant set $B(I)$. This point represents a flat FRW power-law inflationary model [5,17]. The corresponding eigenvalues in the extended state space are (throughout this paper, we shall not explicitly display the corresponding eigenvectors):

$$-\frac{1}{2}(6-k^2), \quad -\frac{1}{2}(6-k^2), \quad -(6-k^2), \quad -(4-k^2), \quad -(2-k^2), \quad -\frac{1}{2}(2-k^2), \quad k^2 - 3\gamma. \tag{3.7}$$

2) $\mathrm{P}^{\pm}_{\mathcal{S}}(VII_h)$: $\Sigma_+ = \tilde{\Sigma} = \Delta = 0, \tilde{A} = \frac{(k^2-2)}{k^2}, N_+ = \pm\frac{\sqrt{l(k^2-2)}}{k}, \Psi = -\frac{\sqrt{2}}{\sqrt{3}k}, \Phi = \frac{2}{\sqrt{3}k}$

These two equilibrium points (the indices "±" correspond to the ± values for N_+), which occur in the Bianchi VII$_h$ invariant set $S(VII_h)$ (since $\tilde{A} \geq 0$, then $k^2 \geq 2$ and therefore $l > 0$), have $q = 0$. These equilibrium points represent an open FRW model [28]. The corresponding eigenvalues in the extended state space are:

$$2 - 3\gamma, \quad -1 \pm \frac{\sqrt{3}i}{k}\sqrt{k^2 - 8/3}, -2 \pm \frac{\sqrt{2}}{k}\sqrt{k^2 - 4(k^2-2)l \pm \sqrt{\left[k^2 - 4(k^2-2)l\right]^2 + 16l\left(k^2-2\right)^2 + k^4}}. \tag{3.8}$$

2a) $\mathrm{P}_{\mathcal{S}}(V)$: $\Sigma_+ = \tilde{\Sigma} = \Delta = 0, \tilde{A} = \frac{(k^2-2)}{k^2}, N_+ = 0, \Psi = -\frac{\sqrt{2}}{\sqrt{3}k}, \Phi = \frac{2}{\sqrt{3}k}$

This case corresponds to points 2) for $l = 0$ and belongs to the set $S(V)$. The corresponding eigenvalues in the extended state space are:

$$2 - 3\gamma, \quad -1 \pm \frac{\sqrt{3}i}{k}\sqrt{k^2 - 8/3}, \quad -2, \quad -2, \quad 0, \quad -4. \tag{3.9}$$

3) $\mathrm{P}^{\pm}_{\mathcal{S}}(II)$: $\Sigma_+ = -\frac{k^2-2}{k^2+16}, \tilde{\Sigma} = 3\Sigma_+^2, \Delta = \Sigma_+ N_+, \tilde{A} = 0, N_+ = \pm 3\frac{\sqrt{-(k^2-2)(k^2-8)}}{k^2+16}, \Psi = -\frac{3\sqrt{6}k}{k^2+16}, \Phi = 6\frac{\sqrt{8-k^2}}{k^2+16}$

These two equilibrium points, for which $q = 8(k^2-2)/(k^2+16) > 0$, exist only for $2 \leq k^2 \leq 8$. These two points represent Bianchi type II models analogous to those found in [26]. The corresponding eigenvalues are:

$$12\frac{k^2-2}{k^2+16}, \quad 6\frac{k^2-8}{k^2+16}, \quad 6\frac{k^2-8}{k^2+16}, \quad 3\frac{(k^2-8) \pm \sqrt{(13k^2-32)(k^2-8)}}{k^2+16}, \quad -3\gamma + 18\frac{k^2}{k^2+16}. \tag{3.10}$$

4) $\mathrm{P}_{\mathcal{S}}(VI_h)$: $\Sigma_+ = \frac{-l(k^2-2)}{n}, \tilde{\Sigma} = -3\Sigma_+^2/l, \Delta = 0, \tilde{A} = \frac{9(k^2-2l)(k^2-2)}{n^2}, N_+ = 0, \Psi = \frac{\sqrt{6}k(1-l)}{n}, \Phi = \frac{2\sqrt{3}\sqrt{(k^2-2l)(1-l)}}{n}$, where $n \equiv k^2(l-3) + 4l$. Since $\tilde{\Sigma} > 0$, we have that $l < 0$ and hence this equilibrium point occurs in the Bianchi VI$_h$ invariant sets. The deceleration parameter is given by $q = 2l(k^2-2)/[k^2(l-3)+4l] \geq 0$, where $k^2 \geq 2$, and this point corresponds to a Collins Bianchi type VI$_h$ solution [32]. The corresponding eigenvalues are:

$$\begin{gathered}6\frac{k^2-2l}{[k^2(l-3)+4l]}, \quad -3\gamma - 6\frac{k^2(1-l)}{[k^2(l-3)+4l]}, \\ 3\frac{(k^2-2l) \pm \sqrt{(k^2-2l)^2 + 8l(1-l)(k^2-2)}}{[k^2(l-3)+4l]}, \quad 3\frac{(k^2-2l) \pm \sqrt{(k^2-2l)[(k^2-2l) - 4(1-l)(k^2-2)]}}{[k^2(l-3)+4l]}.\end{gathered} \tag{3.11}$$

Let us next consider *the Massless Scalar Field Invariant Set* $\mathcal{M}$: there is one equilibrium set which generalizes the work in [26] to include scalar fields:

5) $\mathcal{K}_\mathcal{M}$: $\tilde{\Sigma} + \Sigma_+^2 + \Psi^2 = 1, \Delta = \tilde{A} = N_+ = \Phi = 0, \Psi \neq 0$

This paraboloid, for which $q = 2$, generalizes the parabola $\mathcal{K}$ in [26] defined by $\tilde{\Sigma} + \Sigma_+^2 = 1$ to include a massless scalar field, and represents Jacobs' Bianchi type I non-vacuum solutions [32]. However, the eigenvalues are considerably different from those found in [26], and so we list them all here (the variables which define the subspaces in which the corresponding eigendirections reside are included below in curly braces):

$$\begin{array}{cccccc} 2[(1+\Sigma_+) \pm \sqrt{3\tilde{\Sigma}}], & 0, & 0, & 3(2-\gamma), & 4(1+\Sigma_+), & \frac{\sqrt{6}}{2}\left(\sqrt{6}+k\Psi\right). \\ \{\Delta, N_+\} & \{\Sigma_+, \tilde{\Sigma}\} & \{\Sigma_+, \tilde{\Sigma}, \Psi\} & \{\Sigma_+, \tilde{\Sigma}, \Psi\} & \{\Sigma_+, \tilde{\Sigma}, \tilde{A}, \Psi\} & \{\tilde{\Sigma}, \Phi\} \end{array} \tag{3.12}$$

B. Perfect Fluid Case, $\Psi = \Phi = 0$

As mentioned earlier, the perfect fluid invariant set $\mathcal{F}$ in which $\Psi = \Phi = 0$ was studied by Hewitt and Wainwright [26]; hence this subsection generalizes their results by including a scalar field with an exponential potential. We shall use their notation to label the equilibrium points/sets. There are five such invariant points/sets. In all of these cases the extra two eigenvalues associated with Ψ and Φ are (respectively)

$$-\frac{3}{2}(2-\gamma) < 0, \quad \frac{3}{2}\gamma > 0. \tag{3.13}$$

1) P(I): $\Sigma_+ = \tilde{\Sigma} = \Delta = \tilde{A} = N_+ = \Psi = \Phi = 0$

This equilibrium point, for which $\Omega = 1$, is a saddle for $2/3 < \gamma < 2$ in $\mathcal{F}$ [26] (and is a sink for $0 \leq \gamma < 2/3$), which corresponds to a flat FRW model.

2) $\mathrm{P}^\pm$(II): $\Sigma_+ = -\frac{1}{16}(3\gamma-2), \tilde{\Sigma} = 3\Sigma_+^2, \Delta = \Sigma_+ N_+, \tilde{A} = 0, N_+ = \pm\frac{3}{8}\sqrt{(3\gamma-2)(2-\gamma)}, \Psi = \Phi = 0$

This equilibrium point, for which $\Omega = \frac{3}{16}(6-\gamma)$, is a saddle in the perfect fluid invariant set [26].

3) P(VI_h): $\Sigma_+ = -\frac{1}{4}(3\gamma-2), \tilde{\Sigma} = -3\Sigma_+^2/l, \Delta = 0, \tilde{A} = -\frac{9}{16l}(3\gamma-2)(2-\gamma), N_+ = \Psi = \Phi = 0$

Since $\tilde{\Sigma} > 0$ and $\tilde{A} > 0$, this equilibrium point occurs in the Bianchi VI_h invariant set and corresponds to the Collins solution [32], where $\Omega = \frac{3}{4}(2-\gamma) + \frac{3}{4l}(3\gamma-2)$ (and therefore $2/3 \leq \gamma \leq 2(-l-1)/(3-l)$ and so $l \leq -1$). In [26] this was a sink in $\mathcal{F}$, but is a saddle in the extended state space due to the fact that the two new eigenvalues have values of different sign.

There are also two equilibrium sets, which generalize the work in [26] to include scalar fields:

4) $\mathcal{L}_l^\pm$: $\tilde{\Sigma} = -\Sigma_+(1+\Sigma_+), \Delta = 0, \tilde{A} = (1+\Sigma_+)^2, N_+ = \pm\sqrt{(1+\Sigma_+)[l(1+\Sigma_+) - 3\Sigma_+]}, \Psi = \Phi = 0$

For this set $\Omega = 0$. The local sinks in this set occur when [26]

(a) $l < 0$ (Bianchi type VI_h) for $-\frac{1}{4}(3\gamma-2) < \Sigma_+ < l/(3-l)$ and $l > -(3\gamma-2)/(2-\gamma) < 0$,

(b) $l = 0$ (Bianchi type IV) for $-\frac{1}{4}(3\gamma-2) < \Sigma_+ < 0$,

(c) $l > 0$ (Bianchi type VII_h) for $-\frac{1}{4}(3\gamma-2) < \Sigma_+ < 0$.

The additional two eigenvalues for the full system are:

$$1 - 2\Sigma_+, \quad -2(1+\Sigma_+). \tag{3.14}$$

Finally, let us consider *the Massless Scalar Field Invariant Set* $\mathcal{FM}$:

5) $\mathcal{K}$: $\tilde{\Sigma} + \Sigma_+^2 = 1, \Delta = \tilde{A} = N_+ = \Phi = \Psi = 0$

This parabola, for which $q = 2$, is the special case of $\mathcal{K}_\mathcal{M}$ for which $\Psi = 0$ and corresponds to the parabola $\mathcal{K}$ in [26]. However, the eigenvalues are considerably different from those found in [26] and so we list them all here (the variables define the subspaces in which the corresponding eigendirections reside are included below in curly braces):

$$\begin{array}{cccccc} 2[(1+\Sigma_+) \pm \sqrt{3\tilde{\Sigma}}], & 0, & 0, & 3(2-\gamma), & 4(1+\Sigma_+), & 3. \\ \{\Delta, N_+\} & \{\Sigma_+, \tilde{\Sigma}\} & \{\Psi\} & \{\Sigma_+, \tilde{\Sigma}\} & \{\Sigma_+, \tilde{\Sigma}, \tilde{A}\} & \{\Phi\} \end{array} \tag{3.15}$$

We include here the equilibrium points/sets and corresponding eigenvalues as listed in [26].

Eqm. point/set	Eigenvalues	Comment
$P(I)$	$-\frac{3}{2}(2-\gamma) \quad -3(2-\gamma) \quad (3\gamma-4) \quad \frac{1}{2}(3\gamma-2) \quad \frac{1}{2}(3\gamma-2)$	
$P^{\pm}(II)$	$\frac{3}{4}(3\gamma-2) \quad -\frac{3}{2}(2-\gamma) \quad -\frac{3}{4}(2-\gamma)\left\{1 \pm \sqrt{1-\frac{(3\gamma-2)(6-\gamma)}{2(2-\gamma)}}\right\}$	Constraint eqn. used to eliminate $\tilde{\Sigma}$
$P(VI_h)^{\dagger}$	$-\frac{3}{4}(2-\gamma)(1 \pm \sqrt{1-r^2}) \quad -\frac{3}{4}(2-\gamma)(1 \pm \sqrt{1-q^2})$	Constraint eqn. used to eliminate $\tilde{\Sigma}$
$\mathcal{K}$	$0 \quad 2(1+\Sigma_+) \quad 2(2-\gamma) \quad 2\left[1+\Sigma_+ \pm \sqrt{3(1-\Sigma_+^2)}\right]$	1-D invariant set
$\mathcal{D}$	$0 \quad 0 \quad 2\left[1+\Sigma_+ \pm \sqrt{3\tilde{\Sigma}}\right] \quad 2(1+\Sigma_+)$	2-D invariant set, $\gamma=2$
$\mathcal{L}_l$	$0 \quad -4\Sigma_+ - (3\gamma-2) \quad -2\left[(1+\Sigma_+) \pm 2iN_+\right]$	Constraint eqn. used to eliminate $\tilde{\Sigma}$
$\mathcal{F}_l$	$-2 \quad 4 \quad -2 \quad 0 \quad 0$	$l \geq 0$, non-hyperbolic, $\gamma = 2/3$

TABLE IV. *Equilibrium sets found by Hewitt and Wainwright [26], and the corresponding eigenvalues in the extended space. In the table* $r^2 \equiv 2(3\gamma-2)(1-l_c/l)$, $q^2 \equiv 2r^2/(2-\gamma)$ *and* $l_c \equiv -(3\gamma-2)/(2-\gamma)$.

C. Scaling Solutions

Defining

$$\Psi_S \equiv -\sqrt{\frac{3}{2}}\frac{\gamma}{k}, \quad \Phi_S^2 \equiv \frac{3\gamma(2-\gamma)}{2k^2}, \tag{3.16}$$

and recalling that $0 < \gamma < 2$, there are three equilibrium points corresponding to scaling solutions. Because the scalar field mimics the perfect fluid with the exact same equation of state ($\gamma_\phi = \gamma$) at these equilibrium points, one may combine these two "fluids", via $p_{tot} = p_\phi + p$, $\mu_{tot} = \mu_\phi + \mu$, $p_{tot} = (\gamma-1)\mu_{tot}$; therefore, all of these equilibrium points will correspond to exact perfect fluid models analogous to the equilibrium points found in [26].

The flat isotropic FRW scaling solution [29,30]:
1) $\mathcal{F}_S(I)$: $\Sigma_+ = \tilde{\Sigma} = \Delta = A = N_+ = 0, \Psi = \Psi_S, \Phi = \Phi_S$
The eigenvalues for these points in the extended space, for which $\Omega = 1 - 3\gamma/k^2$ (and therefore $k^2 \geq 3\gamma$) are:

$$-\frac{3}{2}(2-\gamma), \quad -3(2-\gamma), \quad 3\gamma-4, \quad 3\gamma-2, \quad \frac{1}{2}(3\gamma-2), -\frac{3}{4}(2-\gamma) \pm \frac{3}{4}\sqrt{(2-\gamma)(2-9\gamma+24\gamma/k^2)} \tag{3.17}$$

There are two anisotropic scaling solutions:

2) $\mathcal{A}_S(II)$: $\Sigma_+ = -\frac{1}{16}(3\gamma-2), \tilde{\Sigma} = 3\Sigma_+^2, \Delta = \Sigma_+ N_+, \tilde{A} = 0, N_+ = \pm\frac{3}{8}\sqrt{(3\gamma-2)(2-\gamma)}, \Psi = \Psi_S, \Phi = \Phi_S$
The eigenvalues for these points, for which $\Omega = \frac{3}{16}(6-\gamma) - 3\gamma/k^2$ (and therefore $k^2 \geq 16\gamma/[6-\gamma]$), are:

$$\begin{aligned}&\frac{3}{4}\left(3\gamma-2\right), \quad -\frac{3}{2}(2-\gamma),\\ &-\frac{3}{4}\left[(2-\gamma) \pm \sqrt{(2-\gamma)^2 - \frac{3}{4}\left(2-\gamma\right)\left\{2\left(3\gamma-2\right) + \frac{\gamma(6-\gamma)}{k^2}\left(k^2 - \frac{16\gamma}{6-\gamma}\right) \pm \sqrt{E_1}\right\}}\right],\end{aligned} \tag{3.18}$$

where $E_1 \equiv \left[2\left(3\gamma-2\right) - \frac{\gamma(6-\gamma)}{k^2}\left(k^2 - \frac{16\gamma}{6-\gamma}\right)\right]^2 + \frac{8}{9}\left(3\gamma-2\right)\frac{\gamma(6-\gamma)}{k^2}\left(k^2 - \frac{16\gamma}{6-\gamma}\right)$.

3)$\mathcal{A}_S(VI_h)$: $\Sigma_+ = -\frac{1}{4}\left(3\gamma-2\right), \tilde{\Sigma} = -3\Sigma_+^2/l, \Delta = 0, \tilde{A} = -\frac{9}{16l}\left(2-\gamma\right)\left(3\gamma-2\right), N_+ = 0, \Psi = \Psi_S, \Phi = \Phi_S$
These points occur in the Bianchi VI$_h$ invariant set ($l < 0$ since $\tilde{\Sigma} > 0$) for which $\Omega = \frac{3}{4}(2-\gamma) + \frac{3}{4l}(3\gamma-2) - 3\gamma/k^2$ (and therefore $-l^{-1} \leq (2-\gamma)/(3\gamma-2)$ and $k^2 \geq 4\gamma/[(2-\gamma)+(3\gamma-2)/l]$) and correspond to the Collins Bianchi VI$_h$ perfect fluid solutions [32]. The eigenvalues for these equilibrium points are:

$$\begin{aligned}&-\frac{3}{4}\left[(2-\gamma) \pm \sqrt{(2-\gamma)^2 - 4(3\gamma-2)^2\left(\frac{2-\gamma}{3\gamma-2} + \frac{1}{l}\right)}\right],\\ &-\frac{3}{4}\left[(2-\gamma) \pm \sqrt{(2-\gamma)^2 - (2-\gamma)\left[4\gamma\left(1 - \frac{3\gamma}{k^2}\right) + (3\gamma-2)\left(\frac{2-\gamma}{3\gamma-2} + \frac{1}{l}\right) \pm \sqrt{E_2}\right]}\right],\end{aligned} \tag{3.19}$$

where $E_2 \equiv \left[4\gamma\left(1 - \frac{3\gamma}{k^2}\right) - (3\gamma-2)\left(\frac{2-\gamma}{3\gamma-2} + \frac{1}{l}\right)\right]^2 - 128\frac{\gamma^2}{k^2}$.

IV. STABILITY OF THE EQUILIBRIUM POINTS AND SOME GLOBAL RESULTS

The stability of the equilibrium points listed in the previous section can be easily determined from the eigenvalues displayed. Often the stability can be determined by the eigenvalues in the extended state space, otherwise the constraint must be utilized to determine the stability in the six-dimensional state space (i.e., within the constraint surface). In the cases in which this is not possible, we must analyse the eigenvalues in the extended seven-dimensional state-space, and the conclusions that can be drawn are consequently limited. Employing local stability results and utilizing the monotone functions found in Table III, we are able to prove some global results. In the absence of monotone functions, and in the same spirit as Refs. [25] and [26], we conjecture plausible results which are consistent with the local results and the dynamical behaviour on the boundaries and which are substantiated by numerical experiments.

A. The Case $\Omega = 0$

If $\Omega = 0$ and $\Phi = 0$, then the function W_1 in Table III monotonically approaches zero. The existence of the monotone function W_1 implies that the global behaviour of models in the set $\mathcal{M} \cup \mathcal{V}$ can be determined by the local behaviour of the equilibrium points in $\mathcal{M} \cup \mathcal{V}$. Consequently, a portion of the equilibrium sets $\mathcal{K}$ and $\mathcal{K}_{\mathcal{M}}$ (corresponding to local sources) represent the past asymptotic states while the future asymptotic state is represented by $\mathcal{L}_l$, or in the case of Bianchi types I and II, by a point on $\mathcal{K}$.

Therefore, all vacuum models and all massless scalar field models are asymptotic to the past to a Kasner state and are asymptotic to the future either to a plane wave solution (Bianchi types IV, VI_h and VII_h), or to a Kasner state (Bianchi types I and II), or to a Milne state (Bianchi type V).

If $\Omega = 0$ and $\Phi \neq 0$, then the models only contain a scalar field. It was proven in [15] that all Bianchi models evolve to a power-law inflationary state (represented by $\mathrm{P}_{\mathcal{S}}(I)$) when $k^2 < 2$. If $k^2 > 2$, then it was shown in [28] that a subset of Bianchi models of types V and VII_h evolve towards negatively curved isotropic models represented by points $\mathrm{P}_{\mathcal{S}}(V)$ and $\mathrm{P}^{\pm}_{\mathcal{S}}(VII_h)$. In [18] it was shown that when $k^2 > 2$ the future state of a subset of Bianchi type VI_h solutions is represented by the point $\mathrm{P}_{\mathcal{S}}(VI_h)$. It can be seen here that the future state of a subset of Bianchi type II models is represented by the point $\mathrm{P}^{\pm}_{\mathcal{S}}(II)$.

Therefore, all scalar field models with $\Omega = 0$ evolve to a power-law inflationary state if $k^2 < 2$. If $k^2 > 2$, then the future asymptotic state for all Bianchi type IV, V and VII_h models is conjectured to be a negatively–curved, isotropic model and the future asymptotic state for all Bianchi type VI_h models is conjectured to be the Feinstein-Ibáñez anisotropic scalar field model [14]. If $2 < k^2 < 8$, then the future asymptotic state for all Bianchi type II models is the anisotropic Bianchi type II scalar field model, and if $k^2 \geq 8$ then the future asymptotic state is that of a Kasner model. If $2 < k^2 < 6$, then the Bianchi type I models approach a non-inflationary, isotropic (i.e., the point $\mathrm{P}_{\mathcal{S}}(I)$); if $k^2 \geq 6$, then they evolve to a Kasner state in the future.

B. The Case $\Omega \neq 0$, $0 \leq \gamma \leq 2/3$

If $\Omega \neq 0$ and $0 \leq \gamma \leq 2/3$ then the function W_2 in Table III is monotonically decreasing to zero. Therefore, we conclude that the omega-limit set of all non-exceptional orbits (i.e., those orbits excluding equilibrium points, heteroclinic orbits, etc.) of the dynamical system (2.5)-(2.11) is a subset of $S(I)$. This implies that all non-exceptional models with $\Omega \neq 0$ evolve towards the zero–curvature spatially–homogeneous and isotropic models in $S(I)$ and hence isotropize to the future. In [23], it was shown that the zero–curvature spatially–homogeneous and isotropic models evolve towards the power-law inflationary model, represented by the point $\mathrm{P}_{\mathcal{S}}(I)$ when $k^2 < 3\gamma$ or towards the isotropic scaling solution, represented by the point $\mathcal{F}_{\mathcal{S}}(I)$, when $k^2 > 3\gamma$. Using W_1, we also conclude that the past asymptotic state(s) of all non-exceptional models (including models in $S(I)$) is characterized by $\Omega = 0$. In other words, matter is dynamically *unimportant* as these models evolve to the past. It was shown in [23] that all models evolve in the past to some portion of $\mathcal{K}$ or $\mathcal{K}_{\mathcal{M}}$ (the Kasner models) which are local sources.

C. The case $\Omega \neq 0$, $\frac{2}{3} < \gamma < 2$

The following table lists the local sinks for $\frac{2}{3} < \gamma < 2$.

Sink	Bianchi type	k	Other constraints
$\mathrm{P}_{\mathcal{S}}(I)$	I	$k^2 \leq 2$	
$\mathrm{P}_{\mathcal{S}}(VII_h)^{\dagger}$	I	$k^2 = 2$	
$\mathrm{P}_{\mathcal{S}}(VI_{-1})$	III	$k^2 \geq 2$	$\gamma > k^2/(k^2+1), \quad l = -1$
$\mathcal{L}_k^{\pm}(VI_{-1})$	III	all	$\gamma > 1, \quad \Sigma_+ = -1/4$
$\mathrm{P}_{\mathcal{S}}(V)$	V	$k^2 \geq 2$	
$\mathrm{P}_{\mathcal{S}}(VI_h)$	VI_h	$k^2 \geq 2$	$\gamma > 2k^2(1-l)/[k^2(l-3)+4l]$
$\mathcal{L}_k^{\pm}(VI_h)$	VI_h	all	$\gamma > 4/3, \quad \Sigma_+ < -1/2$
$\mathcal{A}_{\mathcal{S}}(VI_h)$	VI_h	$k^2 \geq \frac{4\gamma}{[(2-\gamma)+(3\gamma-2)/l]}$	$l \leq \frac{-(3\gamma-2)}{2-\gamma}$
$\mathrm{P}_{\mathcal{S}}^{\pm}(VII_h)$	VII_h	$k^2 \geq 2$	$l > \frac{k^2}{4(k^2-2)(4-k^2)}$ for $2 < k^2 \leq 4$
			$l < \frac{k^2}{4(k^2-2)(k^2-4)}$ for $k^2 > 4$

TABLE V. *This table lists all of the sinks in the various Bianchi invariant sets for $2/3 < \gamma < 2$. A subset of $\mathcal{K}_{\mathcal{M}}$ acts as a source for all Bianchi class B models.* †*Note: in this case $N_+ = 0$ (i.e., $P_{\mathcal{S}}$=$P_{\mathcal{S}}^{\pm}$) and in fact corresponds to a Bianchi I model.*

The function W_3 is monotonically decreasing to zero in $B(I)\backslash S(I)$ and $B(V)\backslash S(V)$. This implies that there do not exist any periodic or recurrent orbits in these sets and, furthermore, the global behaviour of the Bianchi type I and V models can be determined from the local behaviour of the equilibrium points in these sets. We conjecture that there do not exist any periodic or recurrent orbits in the entire phase space for $\gamma > 2/3$, whence it follows that all global behaviour can be determined from Table V.

We note that a subset of $\mathcal{K}_{\mathcal{M}}$ with $(1+\Sigma_+)^2 > 3\tilde{\Sigma}$, $\Psi > -\sqrt{6}/k$ acts as a source for all Bianchi class B models. For $k^2 < 2$, $\mathrm{P}_{\mathcal{S}}(I)$ is the global attractor (sink). From Table V we see that there are unique global attractors (both past and future) in all invariant sets and hence the asymptotic properties are simple to determine. The sinks and sources for a particular Bianchi invariant set, which may appear in that invariant set or on the boundary corresponding to a (lower-dimensional) specialization of that Bianchi type, can be easily determined from Table V and Figure 1 which lists the specializations of the Bianchi class B models [33].

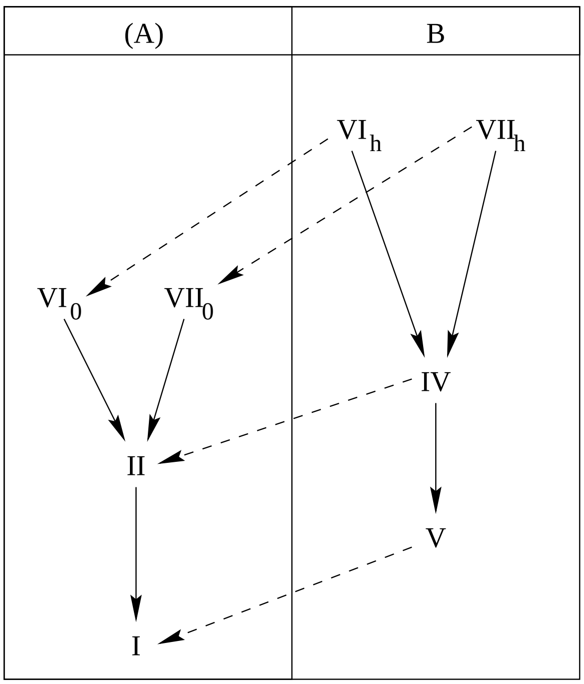

Figure 1: *Specialization diagram for Bianchi class B models obtained by letting a non-zero parameter go to zero. A broken arrow indicates the group class changes (from B to A).*

The most general models are those of Bianchi types VI_h and VII_h. The Bianchi type VII_h models are of particular physical interest since they contain open FRW models as special cases. From Table V and Figure 1 we argue that generically these models (with a scalar field) isotropize to the future, a result which is of great significance. The Bianchi type VI_h models are also of interest since they contain a class of anisotropic scaling solutions that act as attractors for an open set of Bianchi type B models. We note that generically Bianchi type VI_h models do not isotropize for $k^2 \geq 2$.

V. INTERMEDIATE BEHAVIOUR AND THE INVARIANT SET $\mathrm{S(VI_H)}$

It is also of interest to determine the intermediate behaviour of the models. In order to do this, we need to investigate the saddles, determine the dimension of their stable submanifolds, and construct possible heteroclinic sequences. This could then be used, in conjunction with numerical work, to establish the physical properties of the models. For example, we could investigate whether *intermediate isotropization* can occur in Bianchi type VII_h models [34]. There are many different cases to consider depending upon the various bifurcation values and the particular Bianchi invariant set under investigation. As an example, we shall study the heteroclinic sequences in the four-dimensional invariant set

$S(VI_h)$, because it illustrates the method and because such a study emphasizes the importance of anisotropic scaling solutions.

The subspace $S(VI_h)$, which arises from the restrictions $N_+ = \Delta = 0$ and $3\Sigma_+^2 + l\tilde{\Sigma} = 0$ [25] is, in fact, the class of diagonal Bianchi VI_h models and is four–dimensional (and was shown in [24] to illustrate the existence and importance of the anisotropic scaling solutions). From the above restrictions, the system of equations (2.5)-(2.11) now reduce to:

$$\Sigma_+' = (q-2)\Sigma_+ + \tfrac{2}{3}l\tilde{A} \tag{5.1}$$

$$\tilde{A}' = 2(q+2\Sigma_+)\tilde{A} \tag{5.2}$$

$$\Psi' = (q-2)\Psi - \tfrac{1}{2}\sqrt{6}k\Phi^2 \tag{5.3}$$

$$\Phi' = (q+1+\tfrac{1}{2}\sqrt{6}k\Psi)\Phi \tag{5.4}$$

We note that the constraint equation (2.16) is automatically satisfied.

The equilibrium points of the system are a subset of the ones presented in section III and we present those that belong to this subspace and their corresponding eigenvalues in Table VI:

Eqm point/set	Eigenvalues	Stability		Conditions
$\mathcal{L}_k^{\pm}$	$\frac{6}{l-3}$ $\frac{6}{l-3}$ $3\frac{l-1}{l-3}$ $\frac{3}{2}\frac{(3\gamma-2)+l(2-\gamma)}{l-3}$	$s^2 \leq 3\gamma-2:$ $s^2 > 3\gamma-2:$	$W^s:\{\Phi=0\}$ $W^s:\{\Phi=\Omega=0\}$	
$\mathcal{K}_{\mathcal{M}}$	0 $3(2-\gamma)$ $4(1+\Sigma_+)$ $3+\frac{\sqrt{6}}{2}k\sqrt{1+\frac{3-l}{l}\Sigma_+^2}$	$k^2<6$ $k^2 \geq 6$ and $\Sigma_0^2 > \frac{6}{k^2}$ $k^2 \geq 6$ and $\Sigma_0^2 < \frac{6}{k^2}$	$W^s:\{\emptyset\}$ $W^s:\{\emptyset\}$ $W^s:\{\Omega=K=0\}$	
$P(I)$	$-\frac{3}{2}(2-\gamma)$ $-\frac{3}{2}(2-\gamma)$ $3\gamma-2$ $\frac{3}{2}\gamma$		$W^s:\{\Phi=K=0\}$	
$P(VI_h)$	$\frac{3}{2}\gamma$ $-\frac{3}{2}(2-\gamma)$ $-\frac{3}{4}(2-\gamma)(1\pm\sqrt{1-r^2})$		$W^s:\{\Phi=0\}$	$s^2>3\gamma-2$
$P_{\mathcal{S}}(I)$	$\frac{k^2-6}{2}$ $\frac{k^2-6}{2}$ k^2-2 $k^2-3\gamma$	$k^2\leq 2$ $2<k^2<3\gamma$ $k^2>3\gamma$	$W^s:\{S(VI_h)\}$ $W^s:\{K=0\}$ $W^s:\{K=\Omega=0\}$	$k^2<6$
$P_{\mathcal{S}}(VI_h)$	$6\frac{k^2-2l}{[k^2(l-3)+4l]}$ $-3\gamma-6\frac{k^2(1-l)}{[k^2(l-3)+4l]}$ $3\frac{(k^2-2l)\pm\sqrt{(k^2-2l)[(k^2-2l)-4(1-l)(k^2-2)]}}{[k^2(l-3)+4l]}$	$k^2<3\gamma$ or $s^2<s_0^2$ $s^2>s_0^2$	$W^s:\{S(VI_h)\}$ $W^s:\{\Omega=0\}$	$k^2>2$
$\mathcal{F}_S(I)$	$-\frac{3}{2}(2-\gamma)$ $3\gamma-2$ $-\frac{3}{4}(2-\gamma)\pm\sqrt{(2-\gamma)(2-9\gamma+24\gamma^2/k^2)}$		$W^s:\{K=0\}$	$k^2>3\gamma$
$\mathcal{A}_S(VI_h)$	$-\frac{3}{4}(2-\gamma)\pm\sqrt{a\pm\sqrt{b}}$		$W^s:\{S(VI_h)\}$	$s^2>s_0^2$

Table VI: *This table lists all equilibrium points of the system (5.1)-(5.4) and gives the corresponding stable (sub)manifold, W^s. We also give the conditions on the parameters for their existence. We note that $s^2 \equiv \frac{4l}{l-3}$, $s_0^2 \equiv \frac{k^2(3\gamma-2)}{(k^2-3\gamma)}$, $\Sigma_0^2 \equiv 1-\tilde{\Sigma}-\Sigma_+^2 = 1-\frac{l-3}{l}\Sigma_+^2$, $r^2 \equiv 2(3\gamma-2)(1-l_c/l)$, $q^2 \equiv 2r^2/(2-\gamma)$ and $l_c \equiv -(3\gamma-2)/(2-\gamma)$. Note that the four eigenvalues for $\mathcal{A}_S(VI_h)$ are the same as the last four eigenvalues in Eq. (3.19), but are listed here in "short form" for compact notation.*

¿From the eigenvalues in Table V we can study the stability of the equilibrium points and the qualitative behaviour in this four-dimensional subspace. The bifurcation values leading to a change in the asymptotic behaviour are: $k^2=2$, $k^2=6$, $k^2=3\gamma$, $s^2=3\gamma-2$, $s^2=k^2(3\gamma-2)/(k^2-3\gamma)$.

We can see that the possible future asymptotic behaviour is given by points containing a scalar field, either alone or together with a perfect fluid component in the case of the anisotropic scaling solution. These possible future asymptotics are: $P_{\mathcal{S}}(I)$, $P_{\mathcal{S}}(VI_h)$ and $\mathcal{A}_S(VI_h)$.

A. Heteroclinic Sequences

The stable and unstable manifolds of the saddle equilibrium points provide a skeleton of special orbits that play a significant rôle in determining the dynamics of the models (and in particular, their intermediate states) [34]. A (finite) heteroclinic sequence is a set of equilibrium points $E_0, E_1, \ldots, E_n$, where E_0 is a local source, E_n is a local sink and the rest are saddles, such that there is a heteroclinic orbit which joins E_{i-1} to E_i for each $i=1,\ldots,n$ [25]. Below are the heteroclinic orbits in the class of $S(VI_h)$ models for different parameter ranges. Equilibrium points in

parentheses stand for optional intermediate points. We recall that $s^2 \equiv \frac{4l}{l-3}$ and $s_0^2 \equiv \frac{k^2(3\gamma-2)}{(k^2-3\gamma)}$. Finally, when $k^2 > 6$ the Kasner-like ring splits into $\mathcal{K}_{\mathcal{M}}[A]$ (sources) and $\mathcal{K}_{\mathcal{M}}[B]$ (saddles.)

$\boxed{k^2 < 2 < 3\gamma}$

i) $\underline{s^2 < 3\gamma - 2}$:
$\mathcal{K}_{\mathcal{M}} \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}(I)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}(I)$

ii) $\underline{3\gamma - 2 < s^2}$
$\mathcal{K}_{\mathcal{M}} \to \mathcal{L}_k^{\pm} \to P(VI_h) \to P_{\mathcal{S}}(I)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to P(VI_h) \to P_{\mathcal{S}}(I)$

$\boxed{2 < k^2 < 3\gamma}$

i) $\underline{s^2 < 3\gamma - 2}$:
$\mathcal{K}_{\mathcal{M}} \to P(I) \to P_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}^{\pm}(VII_h)$

ii) $\underline{3\gamma - 2 < s^2}$:
$\mathcal{K}_{\mathcal{M}} \to \mathcal{L}_k^{\pm} \to (P(VI_h)) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to P(VI_h) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to P_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h)$

$\boxed{2 < 3\gamma < k^2 < 6}$

i) $\underline{s^2 < 3\gamma - 2 < s_0^2}$:
$\mathcal{K}_{\mathcal{M}} \to P_{\mathcal{S}}(I) \to (\mathcal{F}_{\mathcal{S}}(I)) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to (P(I)) \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to \mathcal{F}_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h)$

ii) $\underline{3\gamma - 2 < s^2 < s_0^2}$:
$\mathcal{K}_{\mathcal{M}} \to \mathcal{L}_k^{\pm} \to (P(VI_h)) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P_{\mathcal{S}}(I) \to (\mathcal{F}_{\mathcal{S}}(I)) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to P(VI_h) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to \mathcal{F}_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h)$

iii) $\underline{3\gamma - 2 < s_0^2 < s^2}$:
$\mathcal{K}_{\mathcal{M}} \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}^{\pm}(VII_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}} \to \mathcal{L}_k^{\pm} \to P(VI_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}} \to P_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}} \to P_{\mathcal{S}}(I) \to \mathcal{F}_{\mathcal{S}}(I) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to P(VI_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}} \to P(I) \to \mathcal{F}_{\mathcal{S}}(I) \to \mathcal{A}_{\mathcal{S}}(VI_h)$

$\boxed{3\gamma < 6 < k^2}$

i) $\underline{s^2 < 3\gamma - 2 < s_0^2}$:
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to (P(I)) \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to P(I) \to \mathcal{F}_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h)$

ii) $\underline{3\gamma - 2 < s^2 < s_0^2}$:
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to \mathcal{L}_k^{\pm} \to (P(VI_h)) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to P(I) \to P(VI_h) \to P_{\mathcal{S}}^{\pm}(VII_h)$
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to P(I) \to \mathcal{F}_{\mathcal{S}}(I) \to P_{\mathcal{S}}^{\pm}(VII_h)$

iii) $\underline{3\gamma - 2 < s_0^2 < s^2}$:
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to \mathcal{L}_k^{\pm} \to P_{\mathcal{S}}^{\pm}(VII_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to \mathcal{L}_k^{\pm} \to P(VI_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to P(I) \to P(VI_h) \to \mathcal{A}_{\mathcal{S}}(VI_h)$
$\mathcal{K}_{\mathcal{M}}[A] \to (\mathcal{K}_{\mathcal{M}}[B]) \to P(I) \to \mathcal{F}_{\mathcal{S}}(I) \to \mathcal{A}_{\mathcal{S}}(VI_h)$

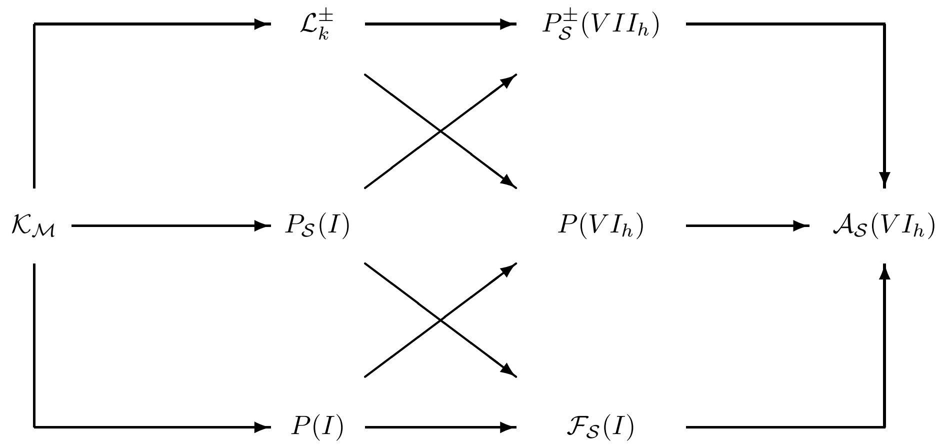

Figure 2. *Heteroclinic sequences in $S(VI_h)$ for $s < 3\gamma < k^2 < 6$ and $3\gamma - 2 < s_0^2 < s^2$, indicating the skeleton of orbits defined by the stable and unstable manifolds of the saddle points. Note that the anisotropic scaling solution $\mathcal{A}_S(VI_h)$ is a stable attractor.*

Figure 2 can be used to construct sequences of orbits joining equilibrium points starting at the past attractor $\mathcal{K}_{\mathcal{M}}$ and ending at the future attractor $\mathcal{A}_S(VI_h)$. For each sequence there will be a family of orbits that shadow this sequence in the state space.

VI. CONCLUSION

In this paper we have discussed the qualitative properties of Bianchi type B cosmological models containing a barotropic fluid and a scalar field with an exponential potential. The most general models are those of type VI$_h$, which include the anisotropic scaling solutions, and those of type VII$_h$, which include the open FRW models.

In cases in which we have been able to find monotone functions we have been able to prove global results. Otherwise, based on the local analysis of the stability of equilibrium points and the dynamics on the boundaries of the appropriate state space, we have presented plausible global results (this is similar to the analysis of perfect fluid models in [25] and [26] in which no monotone functions were found in the Bianchi type VI and VII invariant sets). In all cases, however, our results are further justified by numerical experimentation.

Let us summarize the main results:

- All models with $k^2 < 2$ asymptote toward the flat FRW power-law inflationary model [5,17], corresponding to the global attractor $P_{\mathcal{S}}(I)$, at late times; i.e., all such models isotropize and inflate to the future.
- $F_{\mathcal{S}}(I)$ is a saddle and hence the flat FRW scaling solutions [29,30] do not act as late-time attractors in general [22].
- A subset of $\mathcal{K}_{\mathcal{M}}$ acts as a source for all Bianchi type B models; hence all models are asymptotic in the past to a massless scalar field analogue of the Jacobs anisotropic Bianchi I solutions.
- For $k^2 \geq 2$, Bianchi type VII$_h$ models generically asymptote towards an open FRW scalar field model, represented by one of the local sinks $P_{\mathcal{S}}(V)$ or $P_{\mathcal{S}}^{\pm}(VII_h)$, and hence isotropize to the future.
- For $k^2 \geq 2$, Bianchi type VI$_h$ models generically asymptote towards either an anisotropic scalar field analogue of the Collins solution [32], an anisotropic vacuum solution (with no scalar field) or an anisotropic scaling solution [24], corresponding to the local sinks $P_{\mathcal{S}}^{\pm}(VI_h)$, $\mathcal{L}_k(VI_h)$ or $\mathcal{A}_{\mathcal{S}}(VI_h)$, respectively (depending on the values of a given model's parameters - see table IV for details). These models do not generally isotropize.
- In particular, the equilibrium point $\mathcal{A}_{\mathcal{S}}(VI_h)$ is a local attractor in the Bianchi VI$_h$ invariant set and hence there is an open set of Bianchi type B models containing a perfect fluid and a scalar field with exponential potential which asymptote toward a corresponding anisotropic scaling solution at late times.

We should stress that our analysis and results are applicable to a variety of other cosmological models in, for example, scalar-tensor theories of gravity (which are formally equivalent to general relativity containing a scalar field with an exponential potential) [31,35–37], theories with multiple scalar fields with exponential potentials [38] and string theory [9,39].

In future work we shall study spatially homogeneous models with positive spatial curvature [40] and Bianchi models of type A [41]. However, our ultimate goal is to extend the techniques used in this paper and study the more interesting (and physically more relevant) case of spatially inhomogeneous models.

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